

Zeros of Polynomials

Introduction

If a polynomial can be written as a product of linear factors, then identifying the zeros of the polynomial is straightforward. For example, suppose the polynomial p is defined by

$$p(x) = (x - 3)(x - 1)(x + 7)(2x + 11)$$

The zeros of p are solutions to the equation $p(x) = 0$. The Principle of Zero Products says that if a product is 0, then at least one of the factors must be 0.

Therefore,

$$(x - 3)(x - 1)(x + 7)(2x + 11) = 0$$

$$x - 3 = 0 \text{ or } x - 1 = 0 \text{ or } x + 7 = 0 \text{ or } 2x + 11 = 0$$

$$x = 3 \text{ or } x = 1 \text{ or } x = -7 \text{ or } x = \frac{11}{2}$$

The zeros of p are $x = 3, 1, -7, \frac{11}{2}$.

The Fundamental Theorem of Algebra says that every nonconstant polynomial with real coefficients can be factored into a product of linear and irreducible quadratic factors. An irreducible quadratic has discriminant less than zero, and two complex zeros that occur in conjugate pairs.

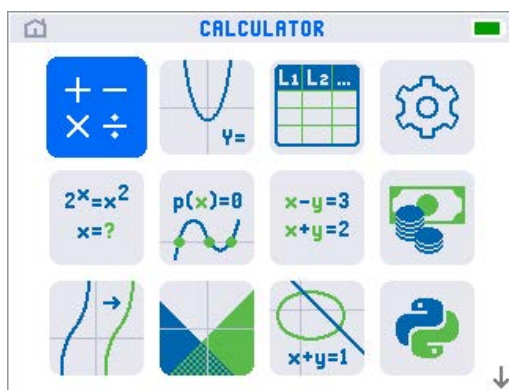
There are several common strategies for factoring a polynomial.

- (1) Factor out a power of x .
- (2) Substitution to reduce the degree of the polynomial.
- (3) Factor by grouping.
- (4) Use a sum or difference formula for powers.
- (5) Long division
- (6) Synthetic division

We also know that a polynomial of degree n has n zeros (real or complex) when counting multiplicities. For each real zero, the polynomial intersects the x -axis. And, it is often important to describe the behavior of the graph of a polynomial near its real zeros. For example, if a is a zero of a polynomial with odd multiplicity, then the graph of the polynomial crosses the x -axis at $(a, 0)$.

Precalculus Using the TI-84 Evo

In this activity, we will use the Polynomial Root Finder App to find the zeros of polynomials. This calculator app is found on the home screen of the TI-84 Evo, , as shown in the figure.



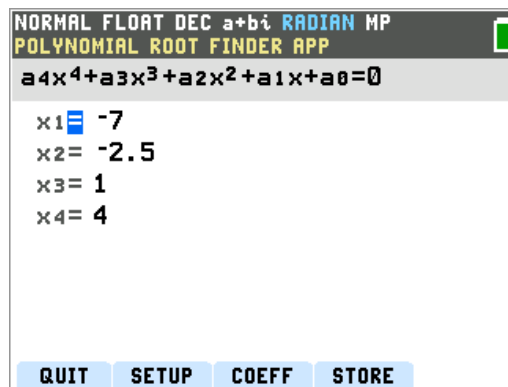
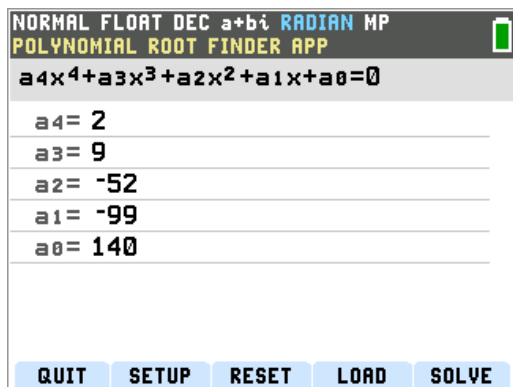
We will also describe the behavior of the graph of polynomials near the real zeros and use technology to confirm the results.

Problem 1: Find all the Zeros of a Polynomial

The polynomial p is defined by $p(x) = 2x^4 + 9x^3 - 52x^2 - 99x + 140$. Find all the zeros of p and sketch a graph of p to confirm your results.

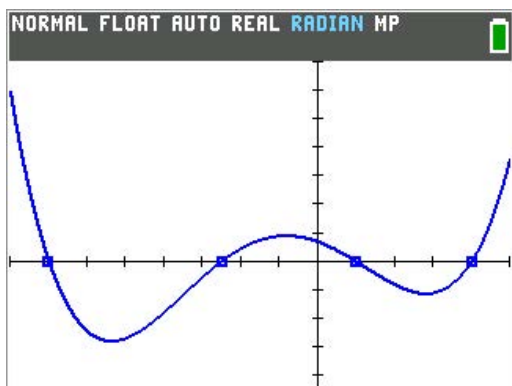
Solution

In the Polynomial Root Finder App, set the degree of the polynomial to 4, enter the coefficients, and solve for the zeros. Remember to use the negative key () rather than the subtraction key () for negative coefficients.



Precalculus Using the TI-84 Evo

The polynomial p has four real roots, $x = -7, -2.5, 1, 4$. The screenshot below shows a graph of the polynomial p and the x -intercepts at the roots.



Using this graph, we can describe the behavior on the polynomial near each root. For example, $p(x) > 0$ for values of x just less than 1 and $p(x) < 0$ for values of x just greater than 1.

Write $p(x)$ in completely factored form, call this function $q(x)$.

Sketch a graph of q . Is it the same as the graph of p ? Explain any differences.

Problem 2: A Polynomial with Complex Roots

The polynomial p is defined by $p(x) = x^4 - 2x^3 + 2x^2 + 38x - 39$.

Find the zeros of p and sketch a graph of p to confirm your results.

Use your graph to describe the behavior of the polynomial near each real root.

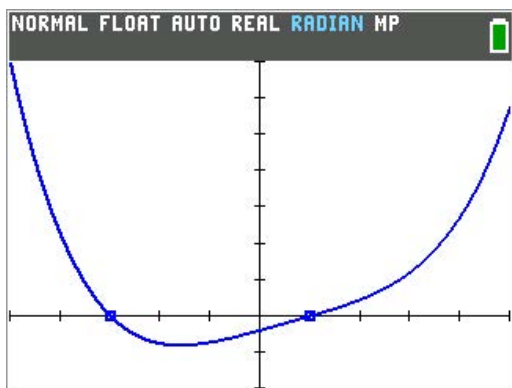
Solution

In the App, set the degree of the polynomial to 4, enter the coefficients, and solve for the zeros.

The polynomial has two real roots and two complex conjugate roots:

$$x = -3, 1, 2 - 3i, 2 + 3i$$

Here is a graph of the polynomial p .



$p(x) > 0$ for values of x just less than -3 and $p(x) < 0$ for values of x just greater than -3 .

$p(x) < 0$ for values of x just less than 1 and $p(x) > 0$ for values of x just greater than 1 .

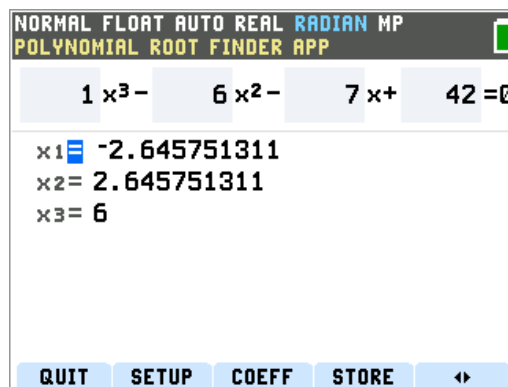
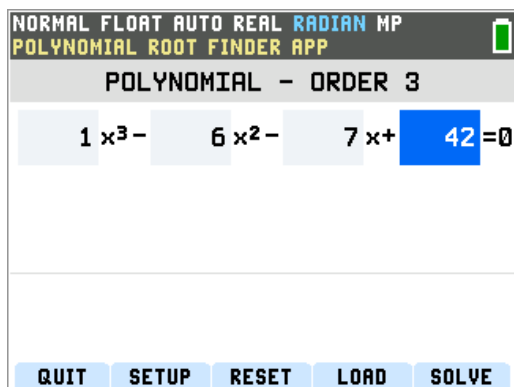
Problem 3: A Polynomial with Irrational Roots

The polynomial p is defined by $p(x) = x^3 - 6x^2 - 7x + 42$.

Find the zeros of p and sketch a graph of p to confirm your results.

Solution

In the Polynomial Root Finder App, set the degree of the polynomial to 3, enter the coefficients, and solve for the zeros.



There is one real root, $x = 6$, and two additional real roots that appear to be additive inverses of one another. Since $x = 6$ is a root of the polynomial, we know that $x - 6$ is a factor. Use long division, or synthetic division, to factor $p(x)$.

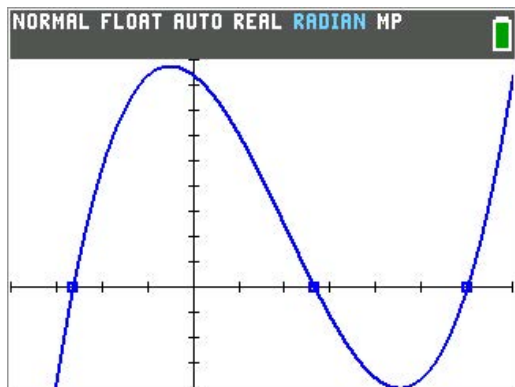
$$p(x) = (x - 6)(x^2 - 7)$$

We can treat the second factor as a difference of two squares.

$$p(x) = (x - 6)(x - \sqrt{7})(x + \sqrt{7})$$

The remaining two roots are $x = \sqrt{7}$ and $x = -\sqrt{7}$.

Here is a graph of the polynomial p .



Problem 4 – Solve an Inequality Involving a Polynomial

Solve the inequality $x^4 - 3x^3 - 31x^2 + 3x + 126 \leq 0$.

Solution

Find the zeros of the polynomial.

Precalculus Using the TI-84 Evo

NORMAL FLOAT DEC a+bi RADIAN MP
POLYNOMIAL ROOT FINDER APP

$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$

$a_4 = 1$
 $a_3 = -3$
 $a_2 = -31$
 $a_1 = 3$
 $a_0 = 126$

QUIT SETUP RESET LOAD SOLVE

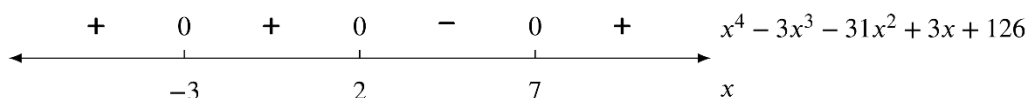
NORMAL FLOAT DEC a+bi RADIAN MP
POLYNOMIAL ROOT FINDER APP

$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$

$x_1 = -3$
 $x_2 = -3$
 $x_3 = 2$
 $x_4 = 7$

QUIT SETUP COEFF STORE

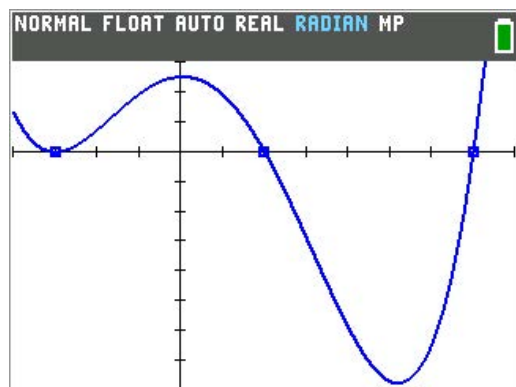
$x = -3$ is a root with multiplicity 2, and $x = 2$ and $x = 7$ are the two remaining roots. Construct a sign chart (or a sign table).



At the endpoints of each interval determined by the sign chart, the polynomial is 0. Therefore, the endpoints are included in the final solution.

Using the sign chart and information about the endpoints, the inequality is satisfied when $2 \leq x \leq 7$. Using interval notation, the solution is $[2, 7]$.

Here is a graph of the polynomial. The solution to the inequality corresponds to the interval in which the graph is on or below the x -axis.



Exercises

(1) Find all the real and complex roots of the polynomial. Sketch the polynomial to confirm your results.

- a. $p(x) = x^3 + 3x^2 - 6x - 8$
- b. $p(x) = x^3 + 15x^2 + 62x + 72$
- c. $p(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$
- d. $p(x) = x^4 - \frac{25x^3}{6} - \frac{101x^2}{3} + \frac{361x}{6} + \frac{770}{3}$
- e. $p(x) = x^4 - \frac{14x^3}{3} - \frac{67x^2}{3} + \frac{350x}{3} - \frac{200}{3}$
- f. $p(x) = x^4 - 5x^3 + x^2 + 5x - 50$
- g. $p(x) = x^4 - 6x^3 - 11x^2 + 74x + 78$
- h. $p(x) = x^4 - 6x^3 - 11x^2 + 216x - 900$
- i. $p(x) = x^4 - 4x^3 + 14x^2 - 4x + 13$
- j. $p(x) = 15x^4 - 29x^3 - 244x^2 + 508x + 240$
- k. $p(x) = x^4 + 3x^3 - 6x^2 - x + 2$
- l. $p(x) = x^5 + 3x^4 - 49x^3 - 97x^2 + 374x + 168$

(2) Solve the inequality involving a polynomial function.

- a. $x^3 - x^2 - 12x \leq 0$
- b. $x^3 - x^2 + \frac{21x}{4} + \frac{29}{4} > 0$
- c. $x^4 - 6x^2 < 0$
- d. $x^4 - 5x^3 - 25x^2 + 65x + 84 \geq 0$
- e. $x^5 - 3x^4 - 18x^3 + 40x^2 > 0$
- f. $x^5 - \frac{23x^4}{5} - \frac{106x^3}{9} + \frac{2438x^2}{45} + \frac{77x}{9} - \frac{1771}{45} < 0$